

LOWER BOUNDS OF MARTINGALE MEASURE DENSITIES IN THE DALANG-MORTON-WILLINGER THEOREM

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ABSTRACT. For a d -dimensional stochastic process $(S_n)_{n=0}^N$ we obtain criteria for the existence of an equivalent martingale measure, whose density z , up to a normalizing constant, is bounded from below by a given random variable f . We consider the case of one-period model ($N = 1$) under the assumptions $S \in L^p$; $f, z \in L^q$, $1/p + 1/q = 1$, where $p \in [1, \infty]$, and the case of N -period model for $p = \infty$. The mentioned criteria are expressed in terms of the conditional distributions of the increments of S , as well as in terms of the boundedness from above of an utility function related to some optimal investment problem under the loss constraints. Several examples are presented.

INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, endowed with a discrete-time filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0}^N$, $\mathcal{F}_N = \mathcal{F}$. Consider a d -dimensional stochastic process $S = (S_n)_{n=0}^N$, adapted to the filtration \mathbb{F} , and a d -dimensional \mathbb{F} -predictable process $\gamma = (\gamma_n)_{n=1}^N$. In the customary securities market model S_n^i describes the discounted price of i th stock and γ_n^i corresponds to the number of stock units in investor's portfolio at time moment n . The gain process is given by

$$G_n^\gamma = \sum_{k=1}^n (\gamma_k, \Delta S_k), \quad \Delta S_k = S_k - S_{k-1}, \quad n = 1, \dots, N, \quad (0.1)$$

where (a, b) is the scalar product of $a, b \in \mathbb{R}^d$.

Let's recall the classical Dalang-Morton-Willinger theorem [3], [13] (ch.V, §2e). As usual, we say that the *No Arbitrage (NA)* condition is satisfied if the inequality $G_N^\gamma \geq 0$ a.s. (with respect to the measure \mathbf{P}) implies that $G_N^\gamma = 0$ a.s. A probability measure \mathbf{Q} on \mathcal{F} is called a *martingale measure* if the process S is a \mathbf{Q} -martingale. The measures \mathbf{P} and \mathbf{Q} are called *equivalent* if their null sets are the same. Denote by $\varkappa_{n-1}(\omega)$ the *support of the regular conditional distribution* $\mathbf{P}_{n-1}(\omega, dx)$ of the random vector ΔS_n with respect to \mathcal{F}_{n-1} .

Theorem 0.1 (Dalang-Morton-Willinger). *The following conditions are equivalent:*

- (i) *NA*;

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- (ii) *there exists an equivalent to \mathbf{P} martingale measure \mathbf{Q} with a.s. bounded density $z = d\mathbf{Q}/d\mathbf{P}$;*
- (iii) *the relative interior of the convex hull of \mathfrak{z}_{n-1} contains the origin a.s., $n = 1, \dots, N$.*

The question concerning the existence of an equivalent martingale measure \mathbf{Q} , whose density z satisfies the lower bound $z \geq c$ (where c is a positive constant) was posed in [8] (Remark 7.5), [4] (Remark 6.5.2). In general, the answer to this question is negative. An evident necessary condition is the integrability of S with respect to \mathbf{P} . Moreover, the example of [4] shows that a measure \mathbf{Q} with the above properties need not exist even for a uniformly bounded process S . A sufficient condition was obtained in [8]. In particular it is satisfied for a process S with independent increments, if the random vectors ΔS_n have finite moments of all orders.

Following [12], let us formulate the problem concerning the existence of an equivalent martingale measure, whose density (up to a normalization constant) is bounded from below by a random variable f , in a more general context. Denote by $\mathbf{E}X$ the expectation with respect to \mathbf{P} , by $L^p = L^p(\mathcal{F}) = L^p(\Omega, \mathcal{F}, \mathbf{P})$, $p \in [1, \infty)$ the Banach spaces of equivalence classes of \mathcal{F} -measurable functions with the norms $\|X\|_p = \mathbf{E}|X|^p$ and by L^∞ the Banach space of essentially bounded functions with the norm $\|X\|_\infty = \text{ess sup}|X|$. The cone L_+^p of non-negative elements induces the partial order on L^p .

Consider the subspace $K \subset L^p$, $p \in [1, \infty)$ of investor's gains (discounted wealth increments). Denote by q the conjugate exponent, that is, $1/p + 1/q = 1$. The condition $K \cap L_+^p = \{0\}$ corresponds to NA. An element $f \in L_+^q$ induces the functional on L^p by the formula $\langle X, f \rangle = \mathbf{E}(Xf)$, $X \in L^p$. It turns out that the existence of an element g , satisfying the conditions

$$\langle X, g \rangle = 0, \quad X \in K; \quad g \geq f, \quad g \in L^q \quad (0.2)$$

is equivalent to the boundedness of f from above on a certain subset K_1 of the subspace K :

$$v_p := \sup_{X \in K_1} \langle X, f \rangle < \infty, \quad K_1 = \{X \in K : \|X^-\|_p \leq 1\}, \quad (0.3)$$

where $X^- = \max\{-X, 0\}$. For $p = \infty$, $q = 1$ this statement is not true in general, see [12], Examples 1 and 3. It becomes true under the assumption that f is bounded from above on the subset $\{X \in K : X^- \in V\}$, where V is a neighborhood of zero in the Mackey topology $\tau(L^\infty, L^1)$, or if L^1 is replaced by the topological dual space $(L^\infty)^*$ of L^∞ . These results are contained in Theorem 1 of [12].

It should be mentioned that the problems, equivalent to (0.3) when $f = 1$, were considered in the recent paper [6]. From the financial point of view they correspond to the maximization of expected gain under the loss constraint, if the loss value is measured either by p th moment $\mathbf{E}|X^-|^p$ for $p \in [1, \infty)$ or by $\text{ess sup}|X^-|$ for $p = \infty$. The equivalence of (0.2) and (0.3) for $p \in (1, \infty)$ follows from the results of the cited paper as well ([6], Theorem 4.1). Unfortunately, the

related statement for $p = \infty$ ([6], Theorem 6.1) is incorrect: a counterexample is, in fact, contained in [12] (Example 3) and its another version is given below (Example 5.4).

Turning back to the finite securities market model, assume that $S \in L^p$ and denote by K the set of random variables G_N^γ , where γ is a *bounded* predictable process. Then the elements g , satisfying (0.2), up to a normalization constant, coincide with the \mathbb{P} -densities of martingale measures: $d\mathbb{Q}/d\mathbb{P} = g/Eg$.

The aim of the present paper is to establish effective criteria for the fulfilment of (0.2), (0.3) for a market model with *finite discrete time* and a *finite collection of stocks*. Such criteria, expressed in terms of the regular conditional distributions of the increments ΔS_n , are obtained for a one-period model under the assumptions $S \in L^p$, $f, g \in L^q$, $p \in [1, \infty]$ (Theorem 1.3), as well as for N -period model in the case $p = \infty$ (Theorem 4.1). These results show also that in the case under consideration the equivalence of (0.2) and (0.3) for $p = \infty$ is nevertheless true! Thereby, we give the negative answer to the question, raised in the end of the paper [12].

In the last part of the paper we give some examples, illustrating the effectiveness of the obtained criteria, and a counterexample to the mentioned statement of [6]. Also, it is interesting to note that the case $p = 1$ of Theorem 1.2 leads to a new proof of the key implication (iii) \implies (ii) of the Dalang-Morton-Willinger theorem (Remark 1.5).

1. ONE-PERIOD MODEL

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let \mathcal{H} be a sub- σ -algebra of \mathcal{F} . A set-valued mapping F , assigning some set $F(\omega) \subset \mathbb{R}^d$ to each $\omega \in \Omega$, is called *\mathcal{H} -measurable*, if $\{\omega : F(\omega) \cap V \neq \emptyset\} \in \mathcal{H}$ for any open set $V \subset \mathbb{R}^d$. A function $\eta : \Omega \mapsto \mathbb{R}^d$ is called a *selector* of F , if $\eta(\omega) \in F(\omega)$ for all $\omega \in \text{dom } F := \{\omega' : F(\omega') \neq \emptyset\}$. An \mathcal{H} -measurable set-valued mapping F with non-empty closed values $F(\omega)$ is measurable if and only if there exists a sequence $(\eta_i)_{i=1}^\infty$ of \mathcal{H} -measurable selectors of F such that the sets $\{\eta_i(\omega)\}_{i=1}^\infty$ are dense in $F(\omega)$ for all ω ([10], Theorem 1B). Such a sequence is called a *Castaing representation* of F .

Denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ -algebra of \mathbb{R}^d . A function $\varphi : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ is called a *Carathéodory function* if (a) $\varphi(\cdot, x) : \Omega \mapsto \mathbb{R}$ is $(\mathcal{H}, \mathcal{B}(\mathbb{R}))$ -measurable for all $x \in \mathbb{R}^d$, (b) $\varphi(\omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous for all $\omega \in \Omega$.

Denote by $L^p(\mathcal{H}, F)$, $1 \leq p < \infty$ the set of equivalence classes of \mathcal{H} -measurable functions η satisfying the conditions $\int |\eta|^p d\mathbb{P} < \infty$, $\eta \in F$ a.s., where $|x| = (x, x)^{1/2}$. We introduce also the sets of equivalence classes of essentially bounded functions $L^\infty(\mathcal{H}, F)$ and of all \mathcal{H} -measurable functions $L^0(\mathcal{H}, F)$, taking values in F a.s. In accordance with the above notation we put $L^p(\mathcal{H}) = L^p(\mathcal{H}, \mathbb{R})$. By $L_+^p(\mathcal{H})$ and $L_{++}^p(\mathcal{H})$ we denote the sets of non-negative and strictly positive elements of $L^p(\mathcal{H})$ respectively. Let $\|X\|_p$ be the norm of an element X of the Banach space $L^p(\mathcal{H})$, $1 \leq p \leq \infty$.

The completion of the σ -algebra \mathcal{H} with respect to the measure \mathbf{P} is denoted by $\mathcal{H}^{\mathbf{P}}$. Note that $L^p(\mathcal{H}^{\mathbf{P}}) = L^p(\mathcal{H})$ in the sense that any $\mathcal{H}^{\mathbf{P}}$ -measurable function possesses an \mathcal{H} -measurable modification.

In the sequel we use the customary notation of convex analysis for the polar $A^\circ = \{x \in \mathbb{R}^d : (x, y) \leq 1, y \in A\}$ of a set $A \subset \mathbb{R}^d$ and also for its Minkowski function and the support function:

$$\mu(x|A) = \inf\{\lambda > 0 : x \in \lambda A\}, \quad s(x|A) = \sup_{y \in A} (x, y).$$

Denote by $\text{conv } A$, $\text{ri } A$ the convex hull and the relative interior of A .

Consider the one-period model (0.1) (that is, $N = 1$). Put $\xi = \Delta S_1$, $\mathcal{H} = \mathcal{F}_0$. Let $\mathbf{P}_\xi(\omega, dx)$ be the regular conditional distribution of ξ with respect to \mathcal{H} and let $\kappa_\xi(\omega)$ be the support of the measure $\mathbf{P}_\xi(\omega, \cdot)$. By $D_\xi(\omega) \subset \mathbb{R}^d$ we denote the linear span of $\kappa_\xi(\omega)$. Define the functions

$$\begin{aligned} \psi_p(\omega, h) &= \left(\int_{\mathbb{R}^d} [(h, x)^-]^p \mathbf{P}_\xi(\omega, dx) \right)^{1/p}, \quad p \in [1, \infty); \\ \psi_\infty(\omega, h) &= s(-h | \kappa_\xi(\omega)) \end{aligned}$$

from $\Omega \times \mathbb{R}^d$ to $[0, \infty]$, and the set-valued mappings

$$\omega \mapsto T_p(\omega) = \{h \in D_\xi(\omega) : \psi_p(\omega, h) \leq 1\}. \quad (1.1)$$

Lemma 1.1. *Assume that $0 \in \text{ri}(\text{conv } \kappa_\xi(\omega))$ a.s. Then T_p is an $\mathcal{H}^{\mathbf{P}}$ -measurable set-valued mapping with a.s. compact values, $p \in [1, \infty]$.*

PROOF. The set-valued mapping $\omega \mapsto \kappa_\xi(\omega)$ is \mathcal{H} -measurable:

$$\{\omega : \kappa_\xi(\omega) \cap V \neq \emptyset\} = \{\omega : \mathbf{P}_\xi(\omega, V) > 0\} \in \mathcal{H}$$

for any open set $V \subset \mathbb{R}^d$. Its values $\kappa_\xi(\omega)$ are closed. It follows from the formula

$$\psi_\infty(\omega, h) = \sup_{i \geq 1} (-h, \eta_i(\omega)),$$

where $(\eta_i)_{i=1}^\infty$ is a Castaing representation of κ_ξ , that the function $\omega \mapsto \psi_\infty(\omega, h)$ is \mathcal{H} -measurable. The same property of ψ_p for $p \in [1, \infty)$ is evident.

Put $\Omega_p = \{\omega : \int |x|^p d\mathbf{P}_\xi(\omega, dx) < \infty\}$ for $p \in [1, \infty)$ and let Ω_∞ be the set of ω , for which the set $\kappa_\xi(\omega)$ is compact. Note that $\Omega_\infty = \{\omega : \sup_{h \in \mathbb{D}} \psi_\infty(\omega, h) < \infty\}$, where \mathbb{D} is a countable dense subset of \mathbb{R}^d . Consequently, $\Omega_p \in \mathcal{H}$, $p \in [1, \infty]$ and $\mathbf{P}(\Omega_p) = 1$. Put $\Omega'_p = \Omega_p \cap \{\omega : 0 \in \text{ri}(\text{conv } \kappa_\xi(\omega))\}$. Clearly, $\Omega'_p \in \mathcal{H}^{\mathbf{P}}$ and $\mathbf{P}(\Omega'_p) = 1$.

Assume that $\omega \in \Omega'_p$. It follows from continuity of ψ_p with respect to h that the set $T_p(\omega)$ is closed. From the condition $0 \in \text{ri}(\text{conv } \kappa_\xi(\omega))$ we see that for $h \in D_\xi(\omega) \setminus 0$ the set $\kappa_\xi(\omega)$ is not contained in the half-space $\{x \in D_\xi(\omega) : (h, x) \geq 0\}$. Therefore, $\psi_p(\omega, h) > 0$, $p \in [1, \infty]$ and the set $T_p(\omega)$ is compact, because $\psi_p(\omega, h) \rightarrow \infty$ when $|h| \rightarrow \infty$, $h \in D_\xi(\omega)$.

Consider the trace of the σ -algebra \mathcal{H} on Ω'_p : $\mathcal{H}_p = \{A \cap \Omega'_p : A \in \mathcal{H}\}$. To complete the proof it is sufficient to check that the set-valued mappings $\omega \mapsto T_p(\omega)$, $\omega \in \Omega'_p$ are \mathcal{H}_p -measurable. We make use of the representation

$T_p(\omega) = \{h \in \mathbb{R}^d : \psi_p(\omega, h) \leq 1\} \cap D_\xi(\omega)$ and the fact that $\psi_p : (\Omega'_p \times \mathbb{R}^d, \mathcal{H}_p \otimes \mathcal{B}(\mathbb{R}^d)) \mapsto [0, \infty)$ are Carathéodory functions. The measurability of the each of set-valued mappings, whose intersection is T_p , follows from Corollary 1Q and Proposition 1H of [10], and the measurability of T_p is implied by Theorem 1M of the same paper. \square

Let us recall the "measurable maximum theorem" ([1], Theorem 18.19).

Lemma 1.2. *Let F be an \mathcal{H} -measurable set-valued mapping with non-empty compact values $F(\omega) \subset \mathbb{R}^d$, and let $\varphi : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ be a Carathéodory function. Put*

$$m(\omega) = \max_{x \in F(\omega)} \varphi(\omega, x), \quad G(\omega) = \{x \in F(\omega) : \varphi(\omega, x) = m(\omega)\}.$$

Then (a) the function m and the set-valued mapping G are \mathcal{H} -measurable; (b) there exists an \mathcal{H} -measurable selector η^ of G .*

Our first main result is the following.

Theorem 1.3. *Let $\xi \in L^p(\mathcal{F}, \mathbb{R}^d)$, $f \in L^q_+(\mathcal{F})$, where $p \in [1, \infty]$ and $1/p + 1/q = 1$. If $0 \in \text{ri}(\text{conv } \kappa_\xi)$ a.s., then the following conditions are equivalent:*

(i) $v_p := \sup\{\mathbb{E}(fX) : \|X^-\|_p \leq 1, X \in K\} < \infty$, where

$$K = \{(\gamma, \xi) : \gamma \in L^\infty(\mathcal{H}, D_\xi)\}; \quad (1.2)$$

(ii) *there exists a random variable $g \in L^q(\mathcal{F})$, satisfying the conditions*

$$\mathbb{E}(g\xi|\mathcal{H}) = 0, \quad g \geq f; \quad (1.3)$$

(iii) $s(a|T_p) \in L^q(\mathcal{H})$, where $a = \mathbb{E}(f\xi|\mathcal{H})$ and T_p is defined by the formula (1.1).

Let us make some remarks before the proof of this theorem (sect. 2 and 3).

Remark 1.4. If $0 \in \text{ri}(\text{conv } \kappa_\xi)$ and $\xi \in L^1(\mathcal{F}, \mathbb{R}^d)$ does not depend on \mathcal{H} , then there exists $g \in L^\infty(\mathcal{F})$:

$$\mathbb{E}(g\xi|\mathcal{H}) = 0, \quad g \geq 1.$$

Actually, in this case $s(a|T_1) = s(\mathbb{E}\xi|T_1)$ does not depend on ω and thus belongs to $L^\infty(\mathcal{H})$.

Remark 1.5. If $0 \in \text{ri}(\text{conv } \kappa_\xi)$ and $\xi \in L^1(\mathcal{F}, \mathbb{R}^d)$, then there exists $g \in L^\infty_{++}(\mathcal{F})$:

$$\mathbb{E}(g\xi|\mathcal{H}) = 0.$$

To prove this statement it is sufficient to note that there exists an \mathcal{H} -measurable function $f \in L^\infty_{++}(\mathcal{H})$ such that

$$s(\mathbb{E}(f\xi|\mathcal{H})|T_1) = s(\mathbb{E}(\xi|\mathcal{H})|T_1)f \in L^\infty(\mathcal{H}).$$

A function $g \in L^\infty(\mathcal{F})$, satisfying (1.3), is the desired one.

In fact, this proves the implication (iii) \implies (ii) of Theorem 0.1 for $N = 1$ and $S \in L^1$. As is known, this is the key point of the proof of the Dalang-Morton-Willinger theorem.

Remark 1.6. Note that

$$a = \mathbb{E}(\mathbb{E}(f|\mathcal{H} \vee \sigma(\xi))\xi|\mathcal{H}) = \int b(\omega, x)x \mathbb{P}_\xi(\omega, dx) \in D_\xi(\omega) \quad \text{a.s.} \quad (1.4)$$

The existence of an $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function $b(\omega, x)$, satisfying the condition $\mathbb{E}(f|\mathcal{H} \vee \sigma(\xi)) = b(\omega, \xi)$, follows from the fact that the σ -algebra $\mathcal{H} \vee \sigma(\xi)$ is generated by the mapping $\omega \mapsto (\omega, \xi(\omega))$ from Ω to the measurable space $(\Omega \otimes \mathbb{R}^d, \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d))$.

Remark 1.7. We have the following convenient representation of the random variable $s(a(\omega)|T_\infty(\omega))$ for $a \in D_\xi$ a.s.:

$$s(a|T_\infty) = \sup\{(h, a) : -h \in D_\xi \cap \mathcal{K}_\xi^\circ\} = s(-a|\mathcal{K}_\xi^\circ) = \mu(-a|\text{conv } \mathcal{K}_\xi) \quad \text{a.s.}$$

It the last equality we have used the formula

$$\mu(x|A^\circ) = \inf\{\lambda > 0 : \lambda^{-1}x \in A^\circ\} = \inf\{\lambda > 0 : s(\lambda^{-1}x|A) \leq 1\} = s(x|A),$$

which is true under the assumption $0 \in A$. We have also used the bipolar theorem: $A^{\circ\circ} = \text{cl}(\text{conv } A)$ and the compactness property of the convex hull of a compact set.

2. PROOF OF THEOREM 1 FOR $p \in [1, \infty)$

Denote by U^p the unit ball of the space $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and put $U_+^p = \{X \in L_+^p : X \in U^p\}$.

Lemma 2.1. *For any element $X \in L^p$, $p \in [1, \infty]$ we have*

$$\|X^+\|_p = \sup\{\langle X, z \rangle : z \in U_+^q\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

PROOF. Consider the elements

$$\zeta_q = \frac{(X^+)^{p/q}}{\|X^+\|_p^{p/q}} \in U_+^q, \quad q \in (1, \infty); \quad \zeta_\infty = I_{\{X \geq 0\}} \in U_+^\infty; \quad \zeta_1^n = \frac{I_{A_n}}{\mathbb{P}(A_n)} \in U_+^1,$$

where $A_n = \{\omega : X(\omega) \geq \|X^+\|_\infty - 1/n\}$. If $X \in L^p$ and q is the conjugate exponent, then

$$\langle X, \zeta_q \rangle = \|X^+\|_p, \quad q \in (1, \infty]; \quad \langle X, \zeta_1^n \rangle \geq \|X^+\|_\infty - \frac{1}{n}.$$

On the other hand,

$$\langle X, z \rangle \leq \langle X^+, z \rangle \leq \|X^+\|_p, \quad z \in U_+^q. \quad \square$$

Though the next result follows from Theorem 1 of [12], it seems convenient to give its direct proof. The idea of this proof is contained also in the paper [11] (Lemma 2.5).

Recall that the closure of a convex set $A \subset L^p$, $p \in [1, \infty)$ in the weak topology $\sigma(L^p, L^q)$, $1/p + 1/q = 1$ coincides with its norm closure in L^p .

Lemma 2.2. *For a subspace $K \subset L^p$, $p \in [1, \infty)$ and an element $f \in L_+^q$, $1/p + 1/q = 1$ the following conditions are equivalent:*

- (a) $\sup_{X \in K_1} \langle X, f \rangle < \infty$, where $K_1 = \{X \in K : \|X^-\|_p \leq 1\}$;
 (b) there exists $g \in L^q$, satisfying the conditions

$$\langle X, g \rangle = 0, \quad X \in K; \quad g \geq f. \quad (2.1)$$

PROOF. (b) \implies (a). If $X \in K_1$ then

$$\langle X, f \rangle = \langle X, g \rangle + \langle X, f - g \rangle = -\langle X, g - f \rangle \leq \langle X^-, g - f \rangle \leq \|g - f\|_q.$$

(a) \implies (b). Put $\lambda = \sup_{X \in K_1} \langle X, f \rangle$. If the assertion (b) is false then

$$(f + \lambda U_+^q) \cap K^\circ = \emptyset, \quad K^\circ = \{z \in L^q : \langle X, z \rangle \leq 0, \quad X \in K\}.$$

By applying the separation theorem ([1], Theorem 5.79) to the $\sigma(L^q, L^p)$ -compact set $f + \lambda U_+^q$ and to the $\sigma(L^q, L^p)$ -closed set K° , we conclude that there exists $Y \in L^p$ such that

$$\sup_{z \in K^\circ} \langle Y, z \rangle < \inf\{\langle Y, \zeta \rangle : \zeta \in f + \lambda U_+^q\}.$$

Since K is a subspace it follows that $\langle Y, z \rangle = 0$, $z \in K^\circ$ and $Y \in K^{\circ\circ} = \text{cl}_p K$ by the bipolar theorem ([1], Theorem 5.103), where $\text{cl}_p K$ is the closure of K in the norm topology of L^p . Moreover,

$$\langle Y, f \rangle + \lambda \inf\{\langle Y, \eta \rangle : \eta \in U_+^q\} > 0. \quad (2.2)$$

By Lemma 2.1 we have

$$\inf\{\langle Y, \eta \rangle : \eta \in U_+^q\} = -\sup\{\langle -Y, \eta \rangle : \eta \in U_+^q\} = -\|Y^-\|_p. \quad (2.3)$$

If $Y^- = 0$ then $\langle Y, f \rangle > 0$ and $\alpha Y \in L_+^p \cap \text{cl}_p K$ for any $\alpha > 0$. Hence, the functional $X \mapsto \langle X, f \rangle$ is unbounded from above on the ray $\{\alpha Y : \alpha > 0\}$, which lies in the set

$$\text{cl}_p K_1 \supset \text{cl}_p \left(\{X : \|X^-\|_p < 1\} \cap K \right) \supset \{X : \|X^-\|_p < 1\} \cap \text{cl}_p K.$$

Here we have used the elementary inclusion $\text{cl}_p(A \cap B) \supset A \cap \text{cl}_p B$, which holds true when the set A is open ([2], chap.1, §1, Proposition 5).

Thus, $\|Y^-\|_p > 0$. It follows from (2.2), (2.3) that

$$\langle Y/\|Y^-\|_p, f \rangle > \lambda$$

in contradiction with the definition of λ since $Y/\|Y^-\|_p \in K_1$. \square

Lemma 2.2 implies that the conditions (i) and (ii) of Theorem 1.3 are equivalent. Indeed, for the subspace (1.2) condition $\langle X, g \rangle = 0$, $X \in K$ means that

$$\mathbf{E}[g(\gamma, \xi)] = \mathbf{E}(\gamma, \mathbf{E}(g\xi|\mathcal{H})) = 0, \quad \gamma \in L^\infty(\mathcal{H}, D_\xi). \quad (2.4)$$

In turn, (2.4) is reduced to the equality $\mathbf{E}(g\xi|\mathcal{H}) = 0$: putting

$$\gamma = \mathbf{E}(g\xi|\mathcal{H})I_{\{|\mathbf{E}(g\xi|\mathcal{H})| \leq M\}} \in L^\infty(\mathcal{H}, D_\xi)$$

and passing in (2.4) to the limit as $M \rightarrow \infty$ we conclude that $\mathbf{E}(g\xi|\mathcal{H}) = 0$ by the monotone convergence theorem.

The equivalence of the conditions (i) and (iii) for all $p \in [1, \infty]$ follows from the equality $v_p = \|s(a|T_p)\|_q$, which is proved in Lemma 2.4 below.

Lemma 2.3. *Let $\xi \in L^0(\mathcal{F}, \mathbb{R}^d)$ and $0 \in \text{ri}(\text{conv } \kappa_\xi)$. If $(\gamma, \xi) \geq 0$ a.s. for some $\gamma \in L^0(\mathcal{H}, D_\xi)$, then $\gamma = 0$ a.s.*

PROOF. Put $A = \{\gamma \neq 0\}$. For any $\omega \in A$ there exists $y \in \kappa_\xi(\omega)$ such that $(\gamma(\omega), y) < 0$ and hence $\int (\gamma(\omega), x)^- P_\xi(\omega, dx) > 0$. If $P(A) > 0$ then we obtain the contradiction:

$$E(\gamma, \xi)^- \geq EE(I_A(\gamma, \xi)^- | \mathcal{H}) = E \left(I_A \int_{\mathbb{R}^d} (\gamma(\omega), x)^- P_\xi(\omega, dx) \right) > 0. \quad \square$$

Lemma 1.1 together with the measurable maximum theorem (Lemma 1.2) imply the existence of an element $h_p^* \in L^0(\mathcal{H}, T_p)$ such that

$$s(a(\omega)|T_p(\omega)) = (h_p^*(\omega), a(\omega)) \quad \text{a.s.}$$

Lemma 2.4. *Under the assumptions of Theorem 1.3 we have*

$$v_p = \sup_{\gamma} \{E(\gamma, a) : \|(\gamma, \xi)^-\|_p \leq 1, \gamma \in L^\infty(\mathcal{H}, D_\xi)\} = \|s(a|T_p)\|_q, \quad p \in [1, \infty].$$

PROOF. (a) The case $1 \leq p < \infty$. Put $U_+^p(\mathcal{H}) = \{g \in L_+^p(\mathcal{H}) : \|g\|_p \leq 1\}$. We have

$$\begin{aligned} U_+^p(\mathcal{F}) &= \{g \in L_+^p(\mathcal{F}) : E(E(g^p | \mathcal{H})) \leq 1\} \\ &= \bigcup_{w \in U_+^p(\mathcal{H})} \{g \in L_+^p(\mathcal{F}) : (E(g^p | \mathcal{H}))^{1/p} \leq w\}. \end{aligned}$$

Consequently,

$$\begin{aligned} v_p &= \sup_{\gamma} \{E(\gamma, a) : (\gamma, \xi)^- \in U_+^p(\mathcal{F}), \gamma \in L^\infty(\mathcal{H}, D_\xi)\} \\ &= \sup_{w \in U_+^p(\mathcal{H})} \sup_{\gamma} \{E(\gamma, a) : (E([(\gamma, \xi)^-]^p | \mathcal{H}))^{1/p} \leq w, \gamma \in L^\infty(\mathcal{H}, D_\xi)\} \end{aligned}$$

On the set $\{w = 0\}$ we have the equality $E([(\gamma, \xi)^-]^p | \mathcal{H}) = 0$. Therefore,

$$E([(\gamma I_{\{w=0\}}, \xi)^-]^p) = 0$$

and $\gamma I_{\{w=0\}} = 0$ by Lemma 2.3. Putting $\gamma = w\theta$, where θ is an \mathcal{H} -measurable vector, we obtain

$$v_p = \sup_{w \in U_+^p(\mathcal{H})} \sup_{\theta} \{Ew(\theta, a) : E([(\theta I_{\{w>0\}}, \xi)^-]^p | \mathcal{H}) \leq 1, w\theta \in L^\infty(\mathcal{H}, D_\xi)\}.$$

Since the values of θ on the set $\{w = 0\}$ do not affect $Ew(\theta, a)$, by the definition of T_p and the equality $E([(\theta, \xi)^-]^p | \mathcal{H}) = \psi_p^p(\omega, \theta(\omega))$ a.s., we get

$$v_p = \sup_{w \in U_+^p(\mathcal{H})} \sup_{\theta} \{Ew(\theta, a) : \theta \in L^0(\mathcal{H}, T_p), w\theta \in L^\infty(\mathcal{H}, D_\xi)\}.$$

But $(\theta, a) \leq s(a|T_p)$ a.s. for $\theta \in L^0(\mathcal{H}, T_p)$. This yields that

$$v_p \leq \sup_{w \in U_+^p(\mathcal{H})} E(s(a|T_p)w) = \|s(a|T_p)\|_q. \quad (2.5)$$

We have used Lemma 2.1 in the last equality.

To obtain the inequality, converse to (2.5), put $\theta = h_p^* I_{\{|w| h_p^*| \leq M\}}$, $M > 0$. Clearly, $w\theta \in L^\infty(\mathcal{H}, D_\xi)$ and

$$\begin{aligned} v_p &\geq \sup_{w \in U_+^p(\mathcal{H})} \mathbb{E}[w(h_p^*, a) I_{\{|w| h_p^*| \leq M\}}] = \sup_{w \in U_+^p(\mathcal{H})} \mathbb{E}(s(a|T_p) w I_{\{|w| h_p^*| \leq M\}}) \\ &= \|s(a|T_p) I_{\{|w| h_p^*| \leq M\}}\|_q. \end{aligned}$$

By the monotone convergence theorem it follows that $v_p \geq \|s(a|T_p)\|_q$.

(b) The case $p = \infty$. It follows from

$$\mathbb{P}((\gamma, \xi) \geq -1) = \mathbb{E}\mathbb{P}(\{(\gamma, \xi) \geq -1\} | \mathcal{H}) = \mathbb{E}\mathbb{P}_\xi(\omega, \{x : (\gamma(\omega), x) \geq -1\})$$

that the condition $\|(\gamma, \xi)^-\|_\infty \leq 1$, meaning that $\mathbb{P}((\gamma, \xi) \geq -1) = 1$, can be represented in the form $\mathbb{P}_\xi(\omega, \{x : (\gamma, x) \geq -1\}) = 1$ a.s. In other words, $\gamma(\omega) \in -\mathcal{K}_\xi^\circ(\omega)$ a.s..

Since $T_\infty = (-\mathcal{K}_\xi^\circ) \cap D_\xi$ this implies that

$$v_\infty = \sup_\gamma \{\mathbb{E}(\gamma, a) : \gamma \in L^\infty(\mathcal{H}, (-\mathcal{K}_\xi^\circ) \cap D_\xi)\} \leq \mathbb{E}s(a|T_\infty).$$

On the other hand, $h_\infty^* I_{\{|h_\infty^*| \leq M\}} \in L^\infty(\mathcal{H}, (-\mathcal{K}_\xi^\circ) \cap D_\xi)$ for all $M > 0$. Therefore,

$$v_\infty \geq \mathbb{E}((h_\infty^*, a) I_{\{|h_\infty^*| \leq M\}}) = \mathbb{E}(s(a|T_\infty) I_{\{|h_\infty^*| \leq M\}})$$

and $v_\infty \geq \mathbb{E}s(a|T_\infty)$ by the monotone convergence theorem. \square

3. PROOF OF THEOREM 1 FOR $p = \infty$

As we have already mentioned, Lemma 2.4 yields that conditions (i) and (iii) of Theorem 1.3 are equivalent. Assume that (ii) is satisfied and put $X = (\gamma, \xi)$, $\gamma \in L^\infty(\mathcal{H}, D_\xi)$. The implication (ii) \implies (i) is a consequence of the inequality

$$\begin{aligned} \mathbb{E}(fX) &= \mathbb{E}(gX) - \mathbb{E}((g - f)X) \leq \mathbb{E}(\gamma, \mathbb{E}(g\xi | \mathcal{H})) + \mathbb{E}((g - f)X^-) \\ &\leq \|g - f\|_1 \|X^-\|_\infty. \end{aligned} \tag{3.1}$$

Let us prove that (ii) follows from (iii). We look for g of the form $g = f + \varphi(\omega, \xi(\omega))$, where $\varphi \in L_+^0(\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d))$. Firstly, the desired function φ should satisfy (1.3):

$$\mathbb{E}(\varphi\xi | \mathcal{H}) = \int \varphi(\omega, x) x \mathbb{P}_\xi(\omega, dx) = -a(\omega) \quad \text{a.s.}$$

Secondly, the function $\omega \mapsto \varphi(\omega, \xi(\omega))$ should be \mathbb{P} -integrable. We construct a function φ with these properties in Lemma 3.3 after some preliminary work.

Lemma 3.1. *Consider a probability measure \mathbb{Q} on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with the support \mathcal{K} . If $0 \in \text{ri}(\text{conv } \mathcal{K})$ then for all y in the linear span D of \mathcal{K} the following equality holds true:*

$$w(y) := \inf \left\{ \int \varphi(x) \mathbb{Q}(dx) : \int \varphi(x) x \mathbb{Q}(dx) = y, \varphi \in L_+^\infty(\mathbb{Q}) \right\} = \mu(y | \text{conv } \mathcal{K}).$$

PROOF. It is easy to check that the epigraph of w : $\text{epi } w = \{(y, \alpha) \in D \times \mathbb{R} : w(y) \leq \alpha\}$ is a convex set (see [9], Lemma 2). Following the general scheme of duality theory (see e.g. [9], [7]) let us find the conjugate function (Young-Fenchel transform) of w :

$$\begin{aligned} w^*(\lambda) &= \sup_{y \in D} \{(y, \lambda) - w(y)\} \\ &= \sup_{\varphi, y} \{(y, \lambda) - \int \varphi(x) \mathbf{Q}(dx) : \int \varphi(x) x \mathbf{Q}(dx) = y, \varphi \in L_+^\infty(\mathbf{Q})\} \\ &= \sup_{\varphi} \left\{ \int \varphi(x) ((x, \lambda) - 1) \mathbf{Q}(dx) : \varphi \in L_+^\infty(\mathbf{Q}) \right\} = \delta(\lambda | \mathfrak{x}^\circ), \quad \lambda \in D. \end{aligned}$$

Here δ is the indicator function: $\delta(\lambda | \mathfrak{x}^\circ) = 0$, $\lambda \in \mathfrak{x}^\circ$; $\delta(\lambda | \mathfrak{x}^\circ) = +\infty$, $\lambda \notin \mathfrak{x}^\circ$. The Young-Fenchel transform of w^* is of the form:

$$w^{**}(y) = \sup_{\lambda \in D} \{(y, \lambda) - w^*(\lambda)\} = s(y | \mathfrak{x}^\circ) = \mu(y | \text{conv } \mathfrak{x}), \quad y \in D.$$

We claim that $\text{dom } w := \{y \in D : w(y) < \infty\} = D$. Clearly, this is the case iff the set $A = \{\int \varphi(x) x \mathbf{Q}(dx) : \varphi \in L_+^\infty(\mathbf{Q})\}$ coincides with D .

Assume that $z \in D$ does not belong to the convex set A . Then there exists a non-zero vector $h \in D$, separating A and z :

$$\left(\int \varphi(x) x \mathbf{Q}(dx), h \right) = \int \varphi(x) (x, h) \mathbf{Q}(dx) \leq (z, h), \quad \varphi \in L_+^\infty(\mathbf{Q}).$$

Putting $\varphi(x) = c I_{\{(h, x) \geq 0\}}$, where $c \in \mathbb{R}_+$, we conclude that the inequality

$$c \int (x, h)^+ \mathbf{Q}(dx) \leq (z, h)$$

should hold true for all $c > 0$. Consequently $(x, h)^+ = 0$ \mathbf{Q} -a.s. Then $(h, x) \leq 0$, $x \in \mathfrak{x}$ and \mathfrak{x} is contained in the subspace orthogonal to h , since $0 \in \text{ri}(\text{conv } \mathfrak{x})$. This means that the linear span of \mathfrak{x} does not coincide with D , a contradiction.

Thus, $\text{dom } w = D$, w is continuous on D and $w = w^{**}$ by the Fenchel-Moreau theorem [7]. \square

Lemma 3.2. *There exists a function $\chi : [0, 1] \times \mathbb{R}^d \mapsto \mathbb{R}$, measurable with respect to $\mathcal{B}([0, 1]) \otimes \mathcal{B}(\mathbb{R}^d)$ and possessing the following property: for any probability measure \mathbf{Q} on $\mathcal{B}(\mathbb{R}^d)$ and for any $\mathcal{B}(\mathbb{R}^d)$ -measurable real-valued function f there exists $r \in [0, 1]$ such that $\chi(r, x) = f(x)$ \mathbf{Q} -a.s.*

Lemma 3.2 is borrowed from the paper [5] (Theorem A.3).

Lemma 3.3. *If $\xi \in L^1(\mathcal{F}, \mathbb{R}^d)$, $0 \in \text{ri}(\text{conv } \mathfrak{x}_\xi)$ a.s., $a \in L^0(\mathcal{H}, D_\xi)$ and $\nu = \mu(-a | \text{conv } \mathfrak{x}_\xi)$, then there exists a function $\varphi \in L_+^0(\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d))$ such that*

$$\begin{aligned} \int \varphi(\omega, x) x \mathbf{P}_\xi(\omega, dx) &= -a(\omega) \text{ a.s.}, \\ \int \varphi(\omega, x) \mathbf{P}_\xi(\omega, dx) &\in [\nu(\omega), \nu(\omega) + \varepsilon(\omega)] \text{ a.s.} \end{aligned}$$

for any \mathcal{H} -measurable function $\varepsilon > 0$.

PROOF. Consider the trace $\mathcal{H}' = \Omega' \cap \mathcal{H}$ of the σ -algebra \mathcal{H} on the set $\Omega' = \{\omega : 0 \in \text{ri}(\text{conv } \kappa(\omega))\} \in \mathcal{H}^P$. Let χ be some function, mentioned in Lemma 3.2. We fix an \mathcal{H} -measurable function $\varepsilon > 0$ and introduce the set-valued mapping $G : \Omega' \mapsto [0, 1]$ by the formula

$$G(\omega) = \{y \in [0, 1] : \int \chi(y, x) P_\xi(\omega, dx) \in [\nu(\omega), \nu(\omega) + \varepsilon(\omega)], \\ \int \chi(y, x)x P_\xi(\omega, dx) = -a(\omega), \int \chi^-(y, x) P_\xi(\omega, dx) = 0\}.$$

Applying Lemma 3.1 to $Q(dx) = P_\xi(\omega, dx)$ and Lemma 3.2, we conclude that $G(\omega) \neq \emptyset$ for all $\omega \in \Omega'$. The functions

$$\int \chi^-(y, x) P_\xi(\omega, dx), \quad \int \chi(y, x) P_\xi(\omega, dx), \quad \int \chi(y, x)x P_\xi(\omega, dx),$$

depending on (ω, y) , are measurable with respect to $\mathcal{H} \otimes \mathcal{B}([0, 1])$: see [3], Lemma 2.2(a). Hence,

$$\text{gr } G = \{(\omega, y) \in \Omega' \times [0, 1] : y \in G(\omega)\} \in \mathcal{H}' \otimes \mathcal{B}([0, 1])$$

and by Aumann's measurable selection theorem there exists an \mathcal{H}' -measurable function $r : \Omega' \mapsto [0, 1]$, satisfying the condition $r(\omega) \in G(\omega)$ a.s. on Ω' ([1], Corollary 18.27). The function $\varphi(\omega, x) = \chi(\hat{r}(\omega), x)$, where \hat{r} is an \mathcal{H} -measurable modification of r , has the desired properties. \square

THE END OF THE PROOF OF THEOREM 1.3. Let us prove that condition (iii) implies (ii) ($p = \infty$). According to the assumption,

$$s(a|T_\infty) = \mu(-a|\text{conv } \kappa_\xi) \in L^1(\mathcal{H}), \quad a = E(f\xi|\mathcal{H}).$$

Let $\varepsilon > 0$ be some constant. Using the notation of Lemma 3.3, we put $g(\omega) = f(\omega) + \varphi(\omega, \xi(\omega))$. The function $g \geq f$ is \mathcal{F} -measurable, P -integrable since

$$E(\varphi \wedge M) = EE(\varphi \wedge M|\mathcal{H}) = E \int (\varphi(\omega, x) \wedge M) P_\xi(\omega, dx) \\ \leq E\mu(-a|\text{conv } \kappa_\xi) + \varepsilon, \quad M > 0, \quad (3.2)$$

and satisfies the equality (1.3):

$$E(g\xi|\mathcal{H}) = a(\omega) + \int \varphi(\omega, x)x P_\xi(\omega, dx) = 0 \quad \text{a.s.} \quad \square$$

4. N -PERIOD MODEL

We turn to N -period market model on a filtered probability space, presented in the introductory section. In addition to the introduced notation denote by $D_{n-1}(\omega)$ the linear span of $\kappa_{n-1}(\omega)$.

Our second main result is the following.

Theorem 4.1. *If the process $S_n \in L^\infty(\mathcal{F}_n, \mathbb{R}^d)$, $n = 0, \dots, N$ satisfies the NA property, then for an element $f \in L^1_{++}(\mathcal{F}, P)$ the following conditions are equivalent:*

- (i) $v := \sup\{\mathbf{E}(fX) : \|X^-\|_\infty \leq 1, X \in K\} < \infty$, where
- $$K = \{G_N^\gamma : \gamma_n \in L^\infty(\mathcal{F}_{n-1}, D_{n-1}), n = 1, \dots, N\};$$
- (ii) there exist an equivalent to \mathbf{P} martingale measure \mathbf{Q} , whose density satisfies the inequality $d\mathbf{Q}/d\mathbf{P} \geq cf$ with some constant $c > 0$;
- (iii) the recurrence relation
- $$\begin{aligned} \beta_N &= f, \\ \beta_n &= \mathbf{E}(\beta_{n+1} | \mathcal{F}_n) + \mu(-a_n | \text{conv } \mathfrak{x}_n), \quad a_n = \mathbf{E}(\beta_{n+1} \Delta S_{n+1} | \mathcal{F}_n) \end{aligned}$$
- specifies the \mathbf{P} -integrable sequence $(\beta_n)_{n=0}^N$.

PROOF. (ii) \implies (i). This statement follows from an estimate, similar to (3.1).
 (i) \implies (iii). Consider the process $X^\gamma = 1 + G^\gamma$:

$$X_{n+1}^\gamma = X_n^\gamma + (\gamma_{n+1}, \Delta S_{n+1}), \quad X_0^\gamma = 1.$$

If the random variable $\beta_n \in L_+^0(\mathcal{F}_n)$ is well-defined, put

$$u_n = \sup_\gamma \{\mathbf{E}(\beta_n X_n^\gamma) : X_k^\gamma \geq 0, \gamma_k \in L^\infty(\mathcal{F}_{k-1}, D_{k-1}), 1 \leq k \leq n\}.$$

By virtue of assumption (i) we have

$$u_N \leq \sup_\gamma \{\mathbf{E}(\beta_N X_N^\gamma) : X_k^\gamma \geq 0, \gamma_k \in L^\infty(\mathcal{F}_{k-1}, D_{k-1}), 1 \leq k \leq n\} = \mathbf{E}f + v < \infty.$$

If $u_{m+1} < \infty$ and the process γ satisfies the conditions of the definition of u_{m+1} , then $\beta_{m+1} \in L^1(\mathcal{F}_{m+1})$ and

$$\mathbf{E}(\beta_{m+1} X_{m+1}^\gamma) = \mathbf{E}(X_m^\gamma \mathbf{E}(\beta_{m+1} | \mathcal{F}_m)) + \mathbf{E}(\gamma_{m+1}, a_m).$$

Consequently,

$$\begin{aligned} u_{m+1} &\geq \mathbf{E}(X_m^\gamma \mathbf{E}(\beta_{m+1} | \mathcal{F}_m)) + t_{m+1}, \\ t_{m+1} &= \sup_{\gamma_{m+1}} \{\mathbf{E}(\gamma_{m+1}, a_m) : X_{m+1}^\gamma \geq 0, \gamma_{m+1} \in L^\infty(\mathcal{F}_m, D_m)\}. \end{aligned} \tag{4.1}$$

The condition $X_{m+1}^\gamma = X_m^\gamma + (\gamma_{m+1}, \Delta S_{m+1}) \geq 0$ a.s. can be rephrased as

$$(\gamma_{m+1}(\omega), x) \geq -X_m^\gamma(\omega), \quad x \in \mathfrak{x}_m(\omega) \text{ a.s.},$$

that is, $\gamma_{m+1} \in -X_m^\gamma \mathfrak{x}_m^\circ$ a.s. (see the proof of Lemma 2.4 for $p = \infty$). Here we take into account that $\gamma_{m+1} = 0$ a.s., if $(\gamma_{m+1}, \Delta S_{m+1}) \geq 0$ and $\gamma_{m+1} \in D_m$ a.s. (Lemma 2.3). Thus,

$$t_{m+1} = \sup_{\gamma_{m+1}} \{\mathbf{E}(\gamma_{m+1}, a_m) : \gamma_{m+1} \in L^\infty(\mathcal{F}_m, -X_m^\gamma \mathfrak{x}_m^\circ)\}.$$

The measurability of the set-valued mapping \mathfrak{x}_m° with respect to \mathcal{F}_m follows from $\mathfrak{x}_m^\circ(\omega) = \bigcap_{i=1}^\infty \{h : (h, \eta_i(\omega)) \leq 1\}$, where $(\eta_i)_{i=1}^\infty$ is a Castaing representation of \mathfrak{x}_m and from Theorem 1M of [10], concerning the measurability of a countable intersection. Owing to the compactness of $\mathfrak{x}_m^\circ(\omega)$ a.s., which follows from $0 \in \text{ri}(\text{conv } \mathfrak{x}_m)$, by the measurable maximum theorem there exists an element $\gamma_{m+1}^* \in L^0(\mathcal{F}_m, -X_m^\gamma \mathfrak{x}_m^\circ)$ such that

$$(\gamma_{m+1}, a_m) \leq (\gamma_{m+1}^*, a_m) = s(a_m | -X_m^\gamma \mathfrak{x}_m^\circ) = X_m^\gamma \mu(-a_m | \text{conv } \mathfrak{x}_m).$$

In particular, $t_{m+1} \leq \mathbb{E}(\gamma_{m+1}^*, a_m)$. On the other hand, by approximation of γ_{m+1}^* by the elements $\gamma_{m+1}^* I_{\{|\gamma_{m+1}^*| \leq M\}} \in L^\infty(\mathcal{F}_m, -X_m^\gamma \mathcal{K}^\circ)$, $M \rightarrow \infty$, we deduce that

$$\mathbb{E}(\gamma_{m+1}^*, a_m) = \lim_{M \rightarrow \infty} \mathbb{E}(\gamma_{m+1}^* I_{\{|\gamma_{m+1}^*| \leq M\}}, a_m) \leq t_{m+1}$$

by the monotone convergence theorem.

By plugging the obtained value $t_{m+1} = \mathbb{E}[X_m^\gamma \mu(-a_m | \text{conv } \mathcal{K}_m)]$ in (4.1), we get

$$v_{m+1} \geq \mathbb{E} \left(\left(\mathbb{E}(\beta_{m+1} | \mathcal{F}_m) + \mu(-a_m | \text{conv } \mathcal{K}_m) \right) X_m^\gamma \right) = \mathbb{E}(\beta_m X_m^\gamma).$$

This inequality holds true under the assumption $X_k^\gamma \geq 0$, $\gamma_k \in L^\infty(\mathcal{F}_{k-1}, D_{k-1})$, $k = 1, \dots, m$. Hence, $v_m \leq v_{m+1} < \infty$. By induction this implies (iii).

(iii) \implies (ii). Put $\nu_n = \mu(-a_n | \text{conv } \mathcal{K}_n)$. Recall that $a_n \in L^0(\mathcal{F}_n, D_n)$ (see (1.4)). By Lemma 3.3 for any $n = 1, \dots, N$ there exists a function $\varphi_n \in L_+^0(\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R}^d))$ such that

$$\int \varphi_n(\omega, x) x \mathbf{P}_n(\omega, dx) = -a_n(\omega) \text{ a.s.}, \quad (4.2)$$

$$\int \varphi_n(\omega, x) \mathbf{P}_n(\omega, dx) \in [\nu_n(\omega), \nu_n(\omega) + \beta_n(\omega)] \text{ a.s.} \quad (4.3)$$

Put $\zeta_{n+1}(\omega) = \varphi_n(\omega, \Delta S_{n+1}(\omega))$. The inequality

$$\mathbb{E}(\zeta_{n+1} \wedge M) = \mathbb{E} \int (\varphi_n(\omega, x) \wedge M) \mathbf{P}_n(\omega, dx) \leq \mathbb{E}(\nu_n + \beta_n),$$

similar to (3.2), these functions are \mathbf{P} -integrable. We can rewrite (4.2), (4.3) as follows:

$$\mathbb{E}(\zeta_{n+1} \Delta S_{n+1} | \mathcal{F}_n) = -a_n, \quad \mathbb{E}(\zeta_{n+1} | \mathcal{F}_n) = \nu_n + \varepsilon_n \beta_n, \quad (4.4)$$

where ε_n is an \mathcal{F}_n -measurable function, taking values in $[0, 1]$. Put $z_N = 1 + \zeta_N/f$,

$$z_n = \frac{1}{1 + \varepsilon_n} \left(1 + \frac{\zeta_n}{\beta_n} \right), \quad n = 1, \dots, N-1; \quad Z = f \prod_{n=1}^N z_n.$$

We claim that the random variable Z is integrable and

$$\mathbb{E}(z_{n+1} \dots z_N f | \mathcal{F}_n) = \beta_n(1 + \varepsilon_n), \quad n = 0, \dots, N-1. \quad (4.5)$$

By virtue of (4.4) and the definition of $(\beta_n)_{n=0}^N$ we have

$$\begin{aligned} \mathbb{E}(z_N f | \mathcal{F}_{N-1}) &= \mathbb{E}(f | \mathcal{F}_{N-1}) + \mathbb{E}(\zeta_N | \mathcal{F}_{N-1}) \\ &= \mathbb{E}(\beta_N | \mathcal{F}_{N-1}) + \nu_{N-1} + \varepsilon_{N-1} \beta_{N-1} = (1 + \varepsilon_{N-1}) \beta_{N-1}. \end{aligned}$$

Assume that the random variable $z_{m+1} \dots z_N f$ is integrable and (4.5) holds true for $n = m$. Then

$$\mathbb{E}(I_{\{z_m \leq M\}} z_m z_{m+1} \dots z_N f) = \mathbb{E}(I_{\{z_m \leq M\}} z_m \beta_m (1 + \varepsilon_m)) \leq \mathbb{E}(\beta_m + \zeta_m).$$

Hence, $z_m z_{m+1} \dots z_N f \in L^1(\mathcal{F})$. Moreover,

$$\begin{aligned} \mathbb{E}(z_m z_{m+1} \dots z_N f | \mathcal{F}_{m-1}) &= \mathbb{E}(z_m \beta_m (1 + \varepsilon_m) | \mathcal{F}_{m-1}) = \mathbb{E}(\beta_m + \zeta_m | \mathcal{F}_{m-1}) \\ &= \mathbb{E}(\beta_m | \mathcal{F}_{m-1}) + \nu_{m-1} + \varepsilon_{m-1} \beta_{m-1} = (1 + \varepsilon_{m-1}) \beta_{m-1}. \end{aligned}$$

By induction (4.5) hold true for all n . In particular, $Z \in L^1(\mathcal{F})$.

Consider a probability measure \mathbf{Q} with the density $d\mathbf{Q}/d\mathbf{P} = cZ$, $c = 1/\mathbf{E}Z$. Evidently, $d\mathbf{Q}/d\mathbf{P} \geq 2^{-N+1}cf$. Let us check that \mathbf{Q} is a martingale measure. Put $A_{n-1} \in \mathcal{F}_{n-1}$. We have

$$\begin{aligned} \frac{1}{c}\mathbf{E}_{\mathbf{Q}}(I_{A_{n-1}}\Delta S_n) &= \mathbf{E}(\mathbf{E}(Z|\mathcal{F}_n)I_{A_{n-1}}\Delta S_n) = \mathbf{E}(z_1 \dots z_n \beta_n (1 + \varepsilon_n) I_{A_{n-1}} \Delta S_n) \\ &= \mathbf{E}(z_1 \dots z_{n-1} I_{A_{n-1}} \mathbf{E}((\beta_n + \zeta_n) \Delta S_n | \mathcal{F}_{n-1})) = 0 \end{aligned}$$

since $\mathbf{E}(\zeta_n \Delta S_n | \mathcal{F}_{n-1}) = -a_{n-1} = -\mathbf{E}(\beta_n \Delta S_n | \mathcal{F}_{n-1})$. \square

5. EXAMPLES

In example 5.1 we concretize the formulas of condition (iii) of Theorem 1.3 for a scalar random variable ξ in the case of general probability space. In example 5.2 we consider a one-period model on a countable space.

Example 5.3 underlines the non-local character of the conditions of Theorem 4.1. Therein we construct a process (S_0, S_1, S_2) with no martingale measure, whose density is bounded from below by a positive constant, but, at the same time, for each of the processes (S_0, S_1) , (S_1, S_2) such a measure exists.

At last, example 5.4 shows that conditions (0.2), (0.3) need not be equivalent for $p = \infty$ even if there exists $z \in L^1_{++}$, satisfying the condition $\mathbf{E}(Xz) = 0$, $X \in K$ and the subspace K is generated by a countable collection of elements.

Example 5.1. Consider the case of scalar random variable ξ . We use the notation of Theorem 1.3. Assume that $\xi \in L^p(\mathcal{F})$, $0 \in \text{ri}(\text{conv } \mathcal{K}_\xi)$ and $f \in L^q_+(\mathcal{F})$, $1/p + 1/q = 1$, $p \in [1, \infty]$.

For $q \in (1, \infty]$ we have

$$\psi_p(\omega, h) = \int [(hx)^-]^p \mathbf{P}_\xi(\omega, dx) = (h^+)^p \mathbf{E}((\xi^-)^p | \mathcal{H})(\omega) + (h^-)^p \mathbf{E}((\xi^+)^p | \mathcal{H})(\omega)$$

and condition (iii) shapes to

$$\begin{aligned} s(a|T_p) &= \sup_h \{ \mathbf{E}(f\xi | \mathcal{H})h : \psi_p(\omega, h) \leq 1 \} \\ &= \frac{(\mathbf{E}(f\xi | \mathcal{H}))^+}{\mathbf{E}((\xi^-)^p | \mathcal{H})^{1/p}} + \frac{(\mathbf{E}(f\xi | \mathcal{H}))^-}{\mathbf{E}((\xi^+)^p | \mathcal{H})^{1/p}} \in L^q(\mathcal{H}). \end{aligned} \quad (5.1)$$

For $q = 1$, $p = \infty$ we have $\text{conv } \mathcal{K}_\xi(\omega) = [\delta_1(\omega), \delta_2(\omega)]$, $0 \in (\delta_1, \delta_2)$ a.s. By virtue of Remark 1.7 condition (iii) becomes

$$\mu(-a|[\delta_1, \delta_2]) = \frac{(\mathbf{E}(f\xi | \mathcal{H}))^+}{|\delta_1|} + \frac{(\mathbf{E}(f\xi | \mathcal{H}))^-}{\delta_2} \in L^1(\mathcal{H}). \quad (5.2)$$

Example 5.2. Here we slightly generalize the model of [4] (Remark 6.5.2), [12] (Example 2). Put $\Omega = \mathbb{N}$. Consider a countable partition $(A_0^j)_{j=1}^\infty$ of the set Ω :

$$\mathbb{N} = \bigcup_{j=1}^\infty A_0^j, \quad A_0^i \cap A_0^k = \emptyset, \quad i \neq k.$$

Denote by \mathcal{H} the σ -algebra, generated by this partition. Let

$$A_0^j = A_1^{2j-1} \cup A_1^{2j}, \quad A_1^{2j-1} \cap A_1^{2j} = \emptyset, \quad j = 1, \dots, \infty$$

and consider the σ -algebra \mathcal{F} generated by the sets $(A_1^j)_{j=1}^\infty$. Assume that $\mathbf{P}(A_1^j) > 0$, $j \in \mathbb{N}$ and let $\xi \in L^p(\mathcal{F})$, $1 \leq p \leq \infty$ be a random variable with $0 \in \text{ri}(\text{conv } \kappa_\xi)$:

$$\xi(\omega) > 0, \quad \omega \in A_1^{2j-1}, \quad \xi(\omega) < 0, \quad \omega \in A_1^{2j}, \quad j \in \mathbb{N}.$$

Let $f \in L_+^q(\mathcal{F})$, $1/p + 1/q = 1$, $p \in [0, \infty]$. For brevity, we put $\eta^j = \eta(\omega)$, $\omega \in A_1^j$ for any \mathcal{F} -measurable random variable η . Define the random variable ρ by the formula

$$\rho(\omega) = \sum_{j=1}^{\infty} \left(f^{2j} \left| \frac{\xi^{2j}}{\xi^{2j-1}} \right| \frac{\mathbf{P}(A_1^{2j})}{\mathbf{P}(A_1^{2j-1})} I_{A_1^{2j-1}}(\omega) + f^{2j-1} \left| \frac{\xi^{2j-1}}{\xi^{2j}} \right| \frac{\mathbf{P}(A_1^{2j-1})}{\mathbf{P}(A_1^{2j})} I_{A_1^{2j}}(\omega) \right).$$

We claim that a necessary and sufficient condition for the existence of a random variable g , satisfying conditions (ii) of Theorem 1.3, is the following:

$$\rho \in L^q(\mathcal{F}). \quad (5.3)$$

We make use of conditions (5.1), (5.2), obtained in example 5.1. In our case

$$\mathbf{E}(f\xi|\mathcal{H})(\omega) = \sum_{j=1}^{\infty} \frac{f^{2j-1}\xi^{2j-1}\mathbf{P}(A_1^{2j-1}) + f^{2j}\xi^{2j}\mathbf{P}(A_1^{2j})}{\mathbf{P}(A_0^j)} I_{A_0^j}(\omega),$$

$$(\mathbf{E}(f\xi|\mathcal{H}))^+(\omega) = \sum_{j=1}^{\infty} \frac{|\xi^{2j}|\mathbf{P}(A_1^{2j})(\rho^{2j} - f^{2j})^+}{\mathbf{P}(A_0^j)} I_{A_0^j}(\omega),$$

$$(\mathbf{E}(f\xi|\mathcal{H}))^-(\omega) = \sum_{j=1}^{\infty} \frac{\xi^{2j-1}\mathbf{P}(A_1^{2j-1})(f^{2j-1} - \rho^{2j-1})^-}{\mathbf{P}(A_0^j)} I_{A_0^j}(\omega).$$

Let $q = 1$. Since $[\delta_1, \delta_2] = \sum_{j=1}^{\infty} [\xi^{2j}, \xi^{2j-1}] I_{A_0^j}$, condition (5.2) shapes to

$$\begin{aligned} \mathbf{E}\mu(-a|[\delta_1, \delta_2]) &= \sum_{j=1}^{\infty} \left((\rho^{2j} - f^{2j})^+ \mathbf{P}(A_1^{2j}) + (f^{2j-1} - \rho^{2j-1})^- \mathbf{P}(A_1^{2j-1}) \right) \\ &= \|(\rho - f)^+\|_1 < \infty, \end{aligned}$$

which is equivalent to (5.3), as long as $f \in L^1(\mathcal{F})$.

For $q \in (1, \infty]$ we use (5.1). By virtue of the equalities

$$\mathbf{E}((\xi^-)^p|\mathcal{H}) = \sum_{j=1}^{\infty} \frac{|\xi^{2j}|^p \mathbf{P}(A_1^{2j})}{\mathbf{P}(A_0^j)} I_{A_0^j}, \quad \mathbf{E}((\xi^+)^p|\mathcal{H}) = \sum_{j=1}^{\infty} \frac{(\xi^{2j-1})^p \mathbf{P}(A_1^{2j-1})}{\mathbf{P}(A_0^j)} I_{A_0^j}$$

we get

$$s(a|T_p) = \sum_{j=1}^{\infty} \frac{(\rho^{2j} - f^{2j})^+ \mathbf{P}(A_1^{2j})^{1-1/p} + (f^{2j-1} - \rho^{2j-1})^- \mathbf{P}(A_1^{2j-1})^{1-1/p}}{\mathbf{P}(A_0^j)^{1-1/p}} I_{A_0^j}.$$

For $q \in (1, \infty)$ condition (5.1) means that

$$\begin{aligned} \|s(a|T_p)\|_q^q &= \sum_{j=1}^{\infty} \left([(\rho^{2j} - f^{2j})^+]^q \mathbf{P}(A_1^{2j}) + [(f^{2j-1} - \rho^{2j-1})^-]^q \mathbf{P}(A_1^{2j-1}) \right) \\ &= \|(\rho - f)^+\|_q^q < \infty, \end{aligned}$$

and is reduced to (5.3). At last, condition $s(a|T_1) \in L^\infty(\mathcal{H})$ for $f \in L^\infty(\mathcal{F})$ is equivalent to the boundedness of ρ .

Example 5.3. Put $\Omega = \mathbb{N}$ and consider the filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$, where the σ -algebra \mathcal{F}_n is generated by the sets $(A_n^j)_{j=1}^\infty$, $n = 0, 1, 2$,

$$A_0^j = \{4j - 3, 4j - 2, 4j - 1, 4j\}, \quad A_1^j = \{2j - 1, 2j\}, \quad A_2^j = \{j\}.$$

Define the probability measure \mathbf{P} on $\mathcal{F}_2 = \mathcal{F}$ by $\mathbf{P}(A_2^{2j-1}) = \mathbf{P}(A_2^{2j}) = 2^{-j-1}$. Note that

$$\begin{aligned} \mathbf{P}(A_1^j) &= \mathbf{P}(A_2^{2j-1}) + \mathbf{P}(A_2^{2j}) = 2^{-j}, \\ \mathbf{P}(A_0^j) &= \mathbf{P}(A_1^{2j-1}) + \mathbf{P}(A_1^{2j}) = 2^{-2j+1} + 2^{-2j} = \frac{3}{2^{2j}}. \end{aligned}$$

We put

$$\begin{aligned} \xi_1(\omega) &= \Delta S_1(\omega) = \sum_{j=1}^{\infty} \left(I_{A_1^{2j-1}}(\omega) - \frac{1}{2^j} I_{A_1^{2j}}(\omega) \right), \\ \xi_2(\omega) &= \Delta S_2(\omega) = \sum_{j=1}^{\infty} \left(I_{A_2^{2j-1}}(\omega) - \frac{1}{2^{j/2}} I_{A_2^{2j}}(\omega) \right). \end{aligned}$$

According to Example 5.2, for the existence of $g_n \in L^1(\mathcal{F}_n)$, $n = 1, 2$, satisfying the conditions

$$\mathbf{E}(g_n \xi_n | \mathcal{F}_{n-1}) = 0, \quad g_n \geq 1,$$

it is necessary and sufficient that the functions

$$\rho_n(\omega) = \sum_{j=1}^{\infty} \left(\frac{1}{2^{j/n}} \frac{\mathbf{P}(A_n^{2j})}{\mathbf{P}(A_n^{2j-1})} I_{A_n^{2j-1}}(\omega) + 2^{j/n} \frac{\mathbf{P}(A_n^{2j-1})}{\mathbf{P}(A_n^{2j})} I_{A_n^{2j}}(\omega) \right), \quad n = 1, 2$$

in the conditions of the form (5.3), are integrable. A simple calculation shows that it is the case:

$$\begin{aligned} \mathbf{E}\rho_1 &= \mathbf{E} \sum_{j=1}^{\infty} \left(\frac{1}{2^{j+1}} I_{A_1^{2j-1}} + 2^{j+1} I_{A_1^{2j}} \right) = \sum_{j=1}^{\infty} \left(\frac{1}{2^{3j}} + \frac{2}{2^j} \right) < \infty, \\ \mathbf{E}\rho_2 &= \mathbf{E} \sum_{j=1}^{\infty} \left(\frac{1}{2^{j/2}} I_{A_2^{2j-1}} + 2^{j/2} I_{A_2^{2j}} \right) = \sum_{j=1}^{\infty} \left(\frac{1}{2^{3j/2+1}} + \frac{1}{2^{j/2+1}} \right) < \infty. \end{aligned}$$

Nevertheless, as we shall see, in the two-period model under consideration, there is no equivalent martingale measure \mathbf{Q} with the density $d\mathbf{Q}/d\mathbf{P} \geq c > 0$, where c is some constant.

Let $\omega \in A_1^j$. With the notation of Theorem 4.1 we have $\beta_2 = 1$,

$$a_1(\omega) = \mathbb{E}(\xi_2 | \mathcal{F}_1)(\omega) = \frac{\mathbb{E}(\xi_2 I_{A_1^j})}{\mathbb{P}(A_1^j)} = \frac{\mathbb{P}(A_2^{2j-1}) - 2^{-j/2} \mathbb{P}(A_2^{2j})}{\mathbb{P}(A_1^j)} = \frac{1 - 2^{-j/2}}{2},$$

$$\mu(-a_1 | \text{conv } \mathcal{K}_1)(\omega) = \inf\{\lambda > 0 : -a_1(\omega) \in \lambda[-2^{-j/2}, 1]\} = 2^{j/2} a_1(\omega),$$

$$\beta_1(\omega) = 1 + \mu(-a_1 | \text{conv } \mathcal{K}_1)(\omega) = 1 + 2^{j/2} \frac{(1 - 2^{-j/2})}{2} = \frac{2^{j/2} + 1}{2}$$

and $\mathbb{E}\beta_1 = \sum_{j=1}^{\infty} (2^{j/2} + 1) \mathbb{P}(A_1^j) / 2 < \infty$.

Now assume that $\omega \in A_0^j$. Then

$$\begin{aligned} a_0(\omega) \mathbb{P}(A_0^j) &= \mathbb{E}(\beta_1 \xi_1 | \mathcal{F}_0)(\omega) \mathbb{P}(A_0^j) = \mathbb{E}(\beta_1 \xi_1 I_{A_0^j}) = \frac{2^{j-1/2} + 1}{2} \mathbb{P}(A_1^{2j-1}) \\ &\quad - \frac{2^j + 1}{2} \frac{1}{2^j} \mathbb{P}(A_1^{2j}) = \frac{1}{2^{2j}} \left(2^{j-1/2} + \frac{1}{2} - \frac{1}{2^{j+1}} \right) \end{aligned}$$

In addition, $a_0(\omega) > 0$ and

$$\mu(-a_0 | \text{conv } \mathcal{K}_0)(\omega) = \inf\{\lambda > 0 : -a_0(\omega) \in \lambda[-2^{-j}, 1]\} = 2^j a_0(\omega).$$

This yields that

$$\mathbb{E}\mu(-a_0 | \text{conv } \mathcal{K}_0) = \sum_{j=1}^{\infty} 2^j a_0^j \mathbb{P}(A_0^j) = \infty, \quad a_0^j = a_0(\omega), \quad \omega \in A_0^j.$$

Therefore, $\beta_0 = \mathbb{E}(\beta_1 | \mathcal{F}_0) + \mu(-a_0 | \text{conv } \mathcal{K}_0) \notin L^1(\mathcal{F}_0)$.

This result shows also that

$$\sup_{\gamma} \{\mathbb{E}G_2^{\gamma} : \gamma_n \in L^{\infty}(\mathcal{F}_{n-1}), \quad n = 1, 2, \quad G_2^{\gamma} \geq -1\} = \infty,$$

whereas

$$\sup_{\gamma_n} \{\mathbb{E}(\gamma_n, \xi_n) : \gamma_n \in L^{\infty}(\mathcal{F}_{n-1}), \quad (\gamma_n, \xi_n) \geq -1\} < \infty, \quad n = 1, 2.$$

Let us present a strategy $\gamma_n \in L^0(\mathcal{F}_{n-1})$, $n = 1, 2$, satisfying the conditions

$$\mathbb{E}G_2^{\gamma} = \infty, \quad G_2^{\gamma} \geq -1.$$

The strategy, constructed below, is "aggressive" and consists in buying of the maximal allowable amount of stocks in each step.

Put $\gamma_1(\omega) = \sum_{j=1}^{\infty} 2^j I_{A_0^j}$. Then

$$G_1^{\gamma} = \sum_{j=1}^{\infty} \left(2^j I_{A_1^{2j-1}} - I_{A_1^{2j}} \right) \geq -1.$$

Since $A_1^{2j-1} = A_2^{4j-3} \cup A_2^{4j-2}$ and

$$\xi_2(\omega) = 1, \quad \omega \in A_2^{4j-3}, \quad \xi_2(\omega) = -\frac{1}{2^{j-1/2}}, \quad \omega \in A_2^{4j-2},$$

we see that the portfolio $\gamma_2(\omega) = \sum_{j=1}^{\infty} 2^{j-1/2}(2^j + 1)I_{A_1^{2j-1}}$ is admissible:

$$G_2^\gamma = \sum_{j=1}^{\infty} \left(2^j I_{A_1^{2j-1}} - I_{A_1^{2j}} \right) + \sum_{j=1}^{\infty} \left(2^{j-1/2}(2^j + 1)I_{A_2^{4j-3}} - (2^j + 1)I_{A_2^{4j-2}} \right) \geq -1$$

and $\mathbf{E}G_2^\gamma = \infty$ as long as

$$\mathbf{P}(A_1^{2j-1}) = 2^{-2j+1}, \quad \mathbf{P}(A_1^{2j}) = 2^{-2j}, \quad \mathbf{P}(A_2^{4j-3}) = \mathbf{P}(A_2^{4j-2}) = 2^{-2j}.$$

Example 5.4. Let $\Omega = \mathbb{N}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let \mathcal{F} be generated by one-point subsets of \mathbb{N} . We put $A_j = \bigcup_{i=j}^{\infty} \{2i\}$, $B_j = \{4j+1\}$,

$$\Delta S_1^j = \xi^j = 2^j I_{B_{j-1}} - I_{A_j}, \quad j \in \mathbb{N}$$

and define the probability measure \mathbf{Q} on \mathcal{F} by $\mathbf{Q}\{2j-1\} = \mathbf{Q}\{2j\} = 2^{-j-1}$. Clearly, \mathbf{Q} is a martingale measure for S :

$$\mathbf{Q}(B_{j-1}) = \mathbf{Q}\{2(2j-1) - 1\} = \frac{1}{2^{2j}}, \quad \mathbf{Q}(A_j) = \sum_{i=j}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^j}$$

$$\mathbf{E}_{\mathbf{Q}} \xi^j = 2^j \mathbf{Q}(B_{j-1}) - \mathbf{Q}(A_j) = 0.$$

Put $B = \bigcup_{j=1}^{\infty} B_{j-1}$ and $B' = \Omega \setminus (A_1 \cup B) = \bigcup_{j=1}^{\infty} \{4j-1\}$. The set Ω coincides with the union of disjoint sets A_1 , B , B' . We note that

$$\mathbf{Q}(A_1) = \frac{1}{2}, \quad \mathbf{Q}(B') = \sum_{j=1}^{\infty} \mathbf{Q}\{2(2j) - 1\} = \sum_{j=1}^{\infty} \frac{1}{2^{2j+1}} = \frac{1}{6}$$

and define an equivalent to \mathbf{Q} "market" measure \mathbf{P} by

$$\mathbf{P}(C) = \mathbf{E}_{\mathbf{Q}}(\zeta I_C), \quad \zeta = \sum_{i=1}^{\infty} 2^{i-1} I_{B_{i-1}} + \frac{3}{4}(I_{A_1} + I_{B'}), \quad C \in \mathcal{F}.$$

Let J be a finite subset of \mathbb{N} . Putting in the inequality

$$G^\gamma(\omega) := \sum_{j \in J} \gamma^j \xi^j(\omega) \geq -1, \quad \omega \in \mathbb{N} \tag{5.4}$$

$\omega = 2m > \max J$ and then $\omega = 4(m-1) + 1$, we get:

$$\sum_{j \in J} \gamma^j \leq 1, \quad 2^m \gamma^m \geq -1.$$

As far as

$$\mathbf{E}_{\mathbf{Q}}(\zeta \xi^j) = \mathbf{E}_{\mathbf{Q}}(2^{2j-1} I_{B_{j-1}} - \frac{3}{4} I_{A_j}) = \frac{1}{2} - \frac{3}{4} \frac{1}{2^j},$$

for γ satisfying (5.4) we have

$$\mathbf{E}G^\gamma = \sum_{j \in J} \gamma^j \mathbf{E}_{\mathbf{Q}}(\zeta \xi^j) = \frac{1}{2} \sum_{j \in J} \gamma^j - \frac{3}{4} \sum_{j \in J} \gamma^j 2^{-j} \leq \frac{1}{2} + \frac{3}{4} \sum_{j=1}^{\infty} \frac{1}{2^{2j}} = \frac{3}{4}.$$

On the other hand, if g is the \mathbf{P} -density of a martingale measure and g is uniformly bounded from below by a constant $c > 0$, then

$$\mathbf{E}(g \xi^j) = 2^j \mathbf{E}(g I_{B_{j-1}}) - \mathbf{E}(g I_{A_j}) = 0,$$

$$\mathbf{E}(gI_{A_j}) \geq c2^j \mathbf{P}(B_{j-1}) = c2^{2j-1} \mathbf{Q}(B_{j-1}) = \frac{c}{2},$$

in contradiction to the dominated convergence theorem, since $\lim_{j \rightarrow \infty} I_{A_j} = 0$ a.s.

Summing up, for the subspace $K \subset L^\infty(\mathcal{F})$, generated by the countable collection of elements $(\xi^j)_{j=1}^\infty$, condition (0.3) is satisfied for $f = 1$, $p = \infty$. Moreover, there exists an element $z = \zeta^{-1} \in L_{++}^1(\mathcal{F})$ such that $\langle X, z \rangle = \mathbf{E}(Xz) = \mathbf{E}_Q X = 0$, $X \in K$. However, there is no element g , satisfying (0.2) for $q = 1$: a counterexample to the assertion of Theorem 6.1 of [6].

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